Numerical range techniques in Quantum information science

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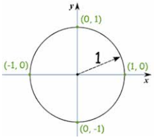
- The numerical range of A can be viewed as a "picture" of the operator A containing useful information of the operator A. Every point $\langle Ax, x\rangle$ is a "pixel" of the picture.

- Note that $W(A) = W(U^*AU)$ if U is unitary.
- If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $W(A) = \{\mu \in \mathbb{C} : |\mu| \leq 1\}$ is the unit disk centered at the origin.



Examples and Convexity

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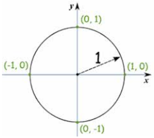

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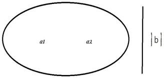

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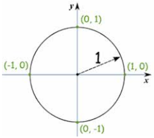

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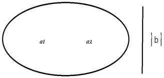

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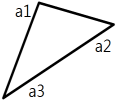

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Then estimate the probability that $A = \lambda I, A = A^*, A \geq 0$, etc.

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- Many theoretical and implementation issues have to be addressed.

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- If $m \geq 4$, then $W(A_1, \ldots, A_m)$ may not be convex even if $\dim H = \infty$.
- Open problem. Characterize $\mathbf{A} = (A_1, \ldots, A_m)$ such that $W(\mathbf{A})$ is convex.

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- (Li, Poon, Wang, 2020) A set $\{A_1, \ldots, A_m\} \subseteq \mathbf{M}_n$ has commuting normal matrices if and only if

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- Current research Generalize the uncertainty principle to multiple observables (Hermitian matrices), and determine the equality case.
- (Li, Poon, Wang, 2020) A set $\{A_1, \ldots, A_m\} \subseteq \mathbf{M}_n$ has commuting normal matrices if and only if there is a positive integer k with $|n/2 - k| \leq 1$ such that the joint k-numerical range

$$W_k(A_1, \ldots, A_m) = \{(\mathrm{tr}\, A_1 P, \ldots, \mathrm{tr}\, A_m P) : P \in \mathbf{D}_n, P = kP^2\}$$

is a polyhedral set. $kP = (kP)^2$ is a rank j Hermitian projection.

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The joint rank p-numerical range of $\mathbf{A} = (A_1, \ldots, A_m)$ is the set $\Lambda_p(\mathbf{A})$ of real m-tuples $(a_1, \ldots, a_m)$ for the existence of a unitary $U \in \mathbf{M}_n$ satisfying

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This concept was introduced in [Choi, Kribs, Zyczkowski, 2006] for the study of quantum error correction schemes.

Quantum error correction

- A quantum channel acting on $\mathbf{D}_n$ is represented by a trace preserving completely positive (TPCP) map $\mathcal{E} : \mathbf{M}_n \to \mathbf{M}_n$ of the form

$$\mathcal{E}(A) = F_1 A F_1^* + \cdots + F_r A F_r^*,$$

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- An error correction code is a subspace $V \subseteq \mathbb{C}^n$ for the existence of a TPCP map (known as the recovery channel) $\mathcal{R}$ such that

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- Then $\mathcal{E}$ has a quantum error correction of dimension p if and only if

$$\Lambda_p(A_1, \ldots, A_m) \neq \emptyset.$$

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- The set may not be convex if $m > 2$.
- Open problem Characterize $\mathbf{A} = (A_1, \ldots, A_m)$ so that $\Lambda_p(\mathbf{A})$ is convex.

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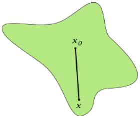

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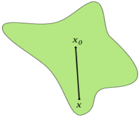

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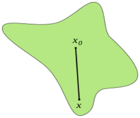

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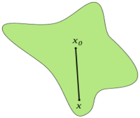

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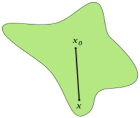

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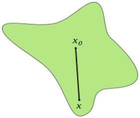

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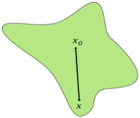

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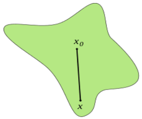

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- Apply the results to quantum information science.

Your comments are most welcome!

Thank you for your attention!